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A new integral representation for the scalar products of Bethe states for the XXX spin chain

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Abstract

Based on the method of separation of variables due to Sklyanin, we construct a new integral representation for the scalar products of the Bethe states for the SU(2) XXX spin 1/2 chain obeying the periodic boundary condition. Due to the compactness of the symmetry group, a twist matrix must be introduced at the boundary in order to extract the separated variables properly. Then by deriving the integration measure and the spectrum of the separated variables, we express the inner product of an on-shell and an off-shell Bethe states in terms of a multiple contour integral involving a product of Baxter wave functions. Its form is reminiscent of the integral over the eigenvalues of a matrix model and is expected to be useful in studying the semi-classical limit of the product.

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1 Introduction

The quantum XXX Heisenberg spin chain is undoubtedly one of the most celebrated quantum integrable systems of fundamental interest. Besides being the prototypical model of the magnetic substance in the condensed matter physics, it has been under intense study as well from the standpoint of mathematical physics of exactly solvable models [1]. As such, innumerable studies have been made in the past and the basic properties of the model are thought to be well understood.

However, about ten years ago this system started to receive a renewed interest from quite a different perspective as it made its appearance in an unexpected way: It was recognized as the precise mathematical structure governing the scaling properties of the

gauge-invariant single-trace composite operators in certain sectors of the $\mathcal{N} = 4$ SU(N) super Yang-Mills theory at the one-loop level [2]. In the simplest situation, where the composite operators are made out of two kinds of adjoint-valued complex scalars Z and X, Z (X) can be identified with the spin up (down) state of an individual spin forming an SU(2) spin chain and the dilatation operator acting on such composite operators takes exactly the form of the well-known XXX spin chain Hamiltonian. This identification allows one to compute the eigenstates and the anomalous dimensions of the composite operators using the techniques of the integrable models, such as the Bethe ansatz [3]. This technique is particularly useful when the number of magnon excitations as well as the number of spins become very large. In such a semi-classical limit, one can recognize the integrable structure quite similar to that of a classical string in a curved spacetime containing an AdS subspace and, as far as the spectrum of the excitation is concerned, this provides a structural parallel strongly suggesting the AdS/CFT correspondence [3].

More recently, further integrable properties of the XXX spin chain have been exploited beyond the spectral level to compute the correlation functions of the composite operators of the super Yang-Mills theory. Namely, in a series of papers [4–8], the method of computing the three point functions $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle$ has been developed, built on earlier works [9–11], in such a way that the three point function can be expressed in terms of the scalar products of the Bethe states of the spin chain Hamiltonian. For relevant configurations, such scalar products can be expressed in terms of the so-called Slavnov determinants [12] [13], which can be further simplified for special cases where BPS operators are involved.

Now in order to study the structural similarity with the three point functions in the strong coupling regime represented by semi-classical string in AdS spacetime[1], it is of importance to consider the semi-classical limit where the number of the magnon excitations as well as spins become very large. For the case where one of the operators is non-BPS and the others are BPS, such a limit has been obtained in a remarkably compact form [6]. More recently, the semi-classical limit of the fully non-BPS three point function has been worked out by Kostov [16, 17]. This was achieved again starting from the expression in terms of the Slavnov's determinant formula in the framework of the algebraic Bethe ansatz, which was quite ingenious.

One of the strong motivations for the study performed in this paper is to construct a representation of the scalar products with which the semi-classical limit may be under-

[1]The one corresponding to the SU(2) sector of the present interest is the string in $AdS_2 \times S^3$. The contribution from the AdS_2 part has been obtained in [14]. The contribution from the S^3 part, which is more involved, will be presented in a forthcoming paper [15].

stood more physically. We believe that such an understanding should be important in order to seek the holographic string picture in the super Yang-Mills theory.

As has been already indicated, studies of the XXX spin 1/2 chain in the past have been performed predominantly in the framework of the algebraic Bethe ansatz. The main ingredients of this method are the elements of the monodromy matrix $\Omega(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$, which are themselves operators acting on the spin chain Hilbert space, with up and down spins at, say, L sites. The operators $A(u) \sim D(u)$ satisfy the Yang-Baxter exchange algebra[2] and from this one sees that $B(u)$ and $C(u)$ can be regarded as the "creation" and the "annihilation" operators respectively, relative to the "vacuum" $|\uparrow^L\rangle$ with all the spins up. Thus one can construct the basic state with M magnon excitations in the form $\prod_{i=1}^M B(u_i)|\uparrow^L\rangle$ and its conjugate $\langle\uparrow^L|\prod_{i=1}^M C(u_i)$. Such a state will be referred to as a Bethe state. It is well-known that a Bethe state becomes an eigenstate of the mutually commuting conserved charges, among which is the Heisenberg Hamiltonian of the spin chain, when the rapidities u_i of the magnons satisfy the Bethe equations, *i.e.* when they are "on-shell". Of fundamental importance in this frame work is the scalar product $\langle\uparrow^L|\prod_{i=1}^M C(v_i)\prod_{i=1}^M B(u_i)|\uparrow^L\rangle$, which can be computed using the Yang-Baxter algebra. When one of the set of rapidities, for instance $\{v_i\}$, are on-shell, the scalar product can be simplified enormously and expressed as a determinant. This is the celebrated Slavnov determinant and practically all the calculations involving the scalar product have been done starting from this expression.

In this article, however, we shall take a different route for the calculation of this scalar product and naturally obtain a different new representation. This alternative is the method of separation of variables (SoV), which was advanced substantially by Sklyanin [18]. The concept of SoV represents the most primitive and fundamental form of integrability, where one reduces the interacting many-body system to mutually decoupled set of dynamical systems, each with a single degree of freedom. Of course the highly non-trivial question is how to actually construct such separated variables $\{x_i\}$ and the corresponding canonically conjugate momenta $\{p_i\}$. For the integrable systems which admit the formulation with Lax operators, Sklyanin proposed a powerful concrete recipe for the construction. Relegating more detailed description to section 3, the prescription applied to the case of XXX spin 1/2 chain says that the solutions x_i of the operator equation $B(x_i) = 0$ provide the separated coordinates, while their conjugate momenta p_i are given essentially by $D(x_i)$. One can indeed check that they satisfy (with appropriate ordering in the quantum case) the canonical Poisson-Lie commutation relations. Therefore if one

[2]We shall only recall a necessary portion of this algebra later when we need them.

can diagonalize and factorize $B(u)$ as $B(u) \propto \prod_{i=1}^{L}(u - x_i)$, with precisely L zeros, x_i provide a complete basis of separated coordinates. Once this is achieved, one can figure out the measure factor $\mu(x_1, \ldots, x_L)$ and compute the scalar products between various states in the x-representation.

Indeed such a method has been applied successfully to some cases where the conventional algebraic Bethe ansatz is not readily applicable. One example is the non-compact SL(2) spin chain in the unitary representation, studied in [19]. One gratifying feature of this case is that in such a unitary representation the hermitian conjugate of $B(u)$ operator is basically itself and hence can be easily diagonalized. The integration measure is found and the scalar product is thus defined in the SoV framework. Another system for which the SoV analysis has been performed is the SU(2) spin chain with anti-periodic boundary condition [20, 21]. In this case, due to the insertion of the twisting matrix $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which flips the spin at the boundary, the operator which should be diagonalized to yield separated variables changes from $B(u)$ to $D(u)$. Since this operator is hermitian and naturally diagonalizable the subsequent analysis à la Sklyanin is straightforward.

Now for the more fundamental case of the SU(2) spin chain with the periodic boundary condition, there are two apparent obstacles in computing the scalar products using the Sklyanin's procedure. The first problem is that because the hermitian conjugate of $B(u)$ is $C(u)$, the basis in which $B(u)$ is diagonal is different from the one in which $C(u)$ is diagonal. Hence the scalar product of our interest $\langle \boldsymbol{v} | \boldsymbol{u} \rangle = \langle \uparrow^L | \prod_{i=1}^{M} C(v_i) \prod_{i=1}^{M} B(u_i) | \uparrow^L \rangle$ cannot be easily computed in $B(u)$-diagonal basis. The second problem is that $B(u)$ operator as it stands is actually not a good operator in the SoV framework, since the coefficient of the highest power u^L in the expansion of $B(u)$ is proportional to $S^- = S^x - iS^y$ belonging to the global SU(2), which is obviously not diagonalizable. It is perhaps for these reasons that this important basic model has not been treated in the SoV basis so far.

We will solve these problems in the following manner. As for the first problem, since we are interested in the case where $\{v_i\}$ are on-shell, we may use the trick due to Kostov and Matsuo [22] to rewrite the scalar product into the form $\sim \langle \downarrow^L | (S_-)^{L-2M} \prod_{i=1}^{M} B(v_i) B(u_i) | \uparrow^L \rangle$, where only the $B(u)$ operators appear. Then, the second problem can be solved by introducing a boundary condition changing twisting matrix $K_\epsilon = \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix}$ so that the modified (regularized) operator $B_\epsilon(u)$ is diagonalizable. As we shall describe in detail in section 4, we can compute the integration measure as well as the wave functions corresponding to the general Bethe states $\prod_{i=1}^{N} B(w_i) | \uparrow^L \rangle$ in the SoV basis. When put together to form the scalar product ϵ-dependence in various quantities cancel precisely. This is as it should be since the original scalar product $\langle \boldsymbol{v} | \boldsymbol{u} \rangle$ is completely well-defined and finite.

The original representation we obtain this way consists of contour integrals over x_i which surround certain poles of the integrand that depend on the index i. This can be recast into a more convenient form where the integration contours for all the x_i's will encircle all the simple poles of the integrand[3]. This expression resembles the integral over the eigenvalues of a matrix model and is expected to be useful in studying the semi-classical limit of the scalar product. In the Appendix A, we shall give a direct independent proof that our integral representation for the $2M = L$ case is equivalent to the Izergin's determinant formula for the so-called domain wall partition function of the 6-vertex model[4]. Using the Kostov-Matsuo trick and taking appropriate limits for the rapidities on both sides, we obtain an alternative proof that our integral representation reproduces the Slavnov's determinant formula for the scalar product of interest.

The rest of this article is organized as follows. We begin in section 2 by giving a brief review of the framework of the algebraic Bethe ansatz where the scalar products of interest are defined and describe several different forms of the determinant formulas for them and related quantities. In section 3 the essence of the Sklyanin's method of separation of variables for classical and quantum integrable models will be summarized. With these preparations, we will derive in section 4 a new integral representation for the scalar product between an on-shell and an off-shell Bethe states for the XXX spin 1/2 chain in a separated variable basis. The remaining problems to be pursued, in particular that of deriving the semi-classical limit from our integral representation, will be briefly discussed in section 5. Two appendices will be provided to give some technical details of the derivation.

2 Algebraic Bethe ansatz and determinant formulas

As a preliminary, we shall give a brief review of the algebraic Bethe ansatz for the XXX spin 1/2 chain and summarize the various known forms of the determinant type formulas for the scalar products between the Bethe states, the quantity of our prime interest. This will at the same time serve to fix our notations.

[3]Multiple integral formulas for the scalar product exist in the literature [23, 24]. Our formula differs from them in form as well as in the context in which it is derived.

[4]Although we will not emphasize it in this article, regarding the XXX spin 1/2 chain from the point of view of the 6-vertex model provides certain useful insights and there have been many interesting works on this topic [13, 25, 26].

2.1 Algebraic Bethe ansatz

The basic ingredient in the framework of the algebraic Bethe ansatz is the so-called Lax operator acting on the product of the spin-chain Hilbert space $\mathcal{H}$ and an auxiliary vector space. In the case of the XXX spin 1/2 chain with L sites, $\mathcal{H}$ is the tensor product of L copies of a two-dimensional vector space, consisting of the up-spin state $|\uparrow\rangle$ and the down-spin state $|\downarrow\rangle$ at each site, and the auxiliary space has the structure of $\mathbb{C}^2$. The Lax operator $L_n(u)$ acting on the n-th site is then given by

$$L_n(u) \equiv u\mathbf{1} + i\sum_{i=x,y,z} S_n^i \sigma^i = \begin{pmatrix} u+iS_n^z & iS_n^- \\ iS_n^+ & u-iS_n^z \end{pmatrix}, \tag{2.1}$$

where S_n^i are the local spin operators[5] and u is the complex spectral parameter. We will impose the periodic boundary condition so that $S_{n+L}^i = S_n^i$. Going around the spin chain, we define the monodromy matrix $\Omega(u)$ as

$$\Omega(u) \equiv L_1(u-\theta_1)\cdots L_L(u-\theta_L) \equiv \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \tag{2.2}$$

$$= u^L\mathbf{1} + iu^{L-1}\left(\sum_{i=x,y,z} S^i\sigma^i + i\sum_{j=1}^{L}\theta_j\right) + \mathcal{O}(u^{L-2})\,. \tag{2.3}$$

Here $S^i = \sum_n S_n^i$ are the total spin operators and we have introduced the inhomogeneity parameters $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_L\}$ at each site, which preserve the integrability. They are necessary for avoiding certain degeneracies in the intermediate steps and are also useful for other purposes[6].

Although the actions of the operators $A(u) \sim D(u)$ on $\mathcal{H}$ are in general quite complicated and non-local, they are known to satisfy rather simple exchange relations, which we call Yang-Baxter algebra [1]. In particular its structure reveals that $B(u)$ and $C(u)$ can be interpreted as a "creation" and an "annihilation" operator respectively with respect to the pseudovacuum $|\uparrow^L\rangle \equiv \underbrace{|\uparrow\rangle \otimes \cdots \otimes |\uparrow\rangle}_{L}$, in which all the spins are up. This allows one to construct the Hilbert space $\mathcal{H}$ as the Fock space spanned by the M-magnon states of the form $|\boldsymbol{u}\rangle = B(u_1)B(u_2)\cdots B(u_M)|\uparrow^L\rangle$, while $C(v)|\uparrow^L\rangle = 0$. u_i's are the rapidities of the magnons, which are related to the momenta by $p = \log\frac{u+i/2}{u-i/2}$. Similarly, the bra states are generated by the operator $C(v)$'s as $\langle\boldsymbol{v}| = \langle\uparrow^L|C(v_1)C(v_2)\cdots C(v_M)$, built upon the

[5] We define $S_n^\pm$ as $S_n^\pm \equiv S^x \pm iS^y$.

[6] Although the physical meaning of the inhomogeneity parameters in the context of the three point functions has not been fully clarified, they are useful in generating loop corrections from the tree-level contributions [7, 8].

dual pseudovacuum $\langle\uparrow^L|$ satisfying $\langle\uparrow^L|B(u)=0$ and $\langle\uparrow^L|\uparrow^L\rangle=1$. These Fock states will be referred to as generic Bethe states.

Of particular importance is the transfer matrix given by $T(u)\equiv \mathrm{Tr}\,\Omega(u)=A(u)+D(u)$, which upon expanded in powers of u generates all the mutually commuting conserved quantities, including the Hamiltonian of the spin chain. The (dual) pseudovacuum is known to be the eigenstate of $T(u)$ in the manner

$$A(u)|\uparrow^L\rangle = Q^+_{\boldsymbol{\theta}}(u)|\uparrow^L\rangle\,, \qquad D(u)|\uparrow^L\rangle = Q^-_{\boldsymbol{\theta}}(u)|\uparrow^L\rangle\,, \tag{2.4}$$

$$\langle\uparrow^L|A(u) = \langle\uparrow^L|Q^+_{\boldsymbol{\theta}}(u)\,, \qquad \langle\uparrow^L|D(u) = \langle\uparrow^L|Q^-_{\boldsymbol{\theta}}(u)\,, \tag{2.5}$$

where $Q_{\boldsymbol{\theta}}$ functions are defined as[7]

$$Q_{\boldsymbol{\theta}}(u)\equiv\prod_{i=1}^{L}(u-\theta_i)\,, \qquad Q^{\pm}_{\boldsymbol{\theta}}(u)\equiv\prod_{i=1}^{L}\left(u-\theta_i\pm\frac{i}{2}\right)\,. \tag{2.6}$$

Using this fact, the action of $T(u)$ on the generic Bethe state $|\boldsymbol{u}\rangle=\prod_{i=1}^{M}B(u_i)|\uparrow^L\rangle$ can be computed by pushing $A(u)$ and $D(u)$ through $B(u_i)$'s using the exchange relations such as $(u-v)A(v)B(u)=(u-v+i)B(u)A(v)-iB(v)A(u)$ and a similar one between $D(v)$ and $B(u)$. One then finds that $|\boldsymbol{u}\rangle$ becomes the eigenstate of $T(u)$ if and only if the following sets of equations, called the Bethe ansatz equations, for the rapidities are satisfied:

$$\prod_{k=1}^{L}\left(\frac{u_i-\theta_k+\frac{i}{2}}{u_i-\theta_k-\frac{i}{2}}\right)=\prod_{j\neq i}^{M}\left(\frac{u_i-u_j+i}{u_i-u_j-i}\right)\,. \tag{2.7}$$

This equation can also be interpreted as a periodicity condition for the phases of the magnon excitations as we go around the chain. When this equation is satisfied, the Bethe state is said to be "on-shell" (otherwise called "off-shell"). In that case, the eigenvalue $t_{\mathbf{u}}(u)$ of the transfer matrix $T(u)$ is given by

$$t_{\boldsymbol{u}}(u)=Q^+_{\boldsymbol{\theta}}(u)\frac{Q^{--}_{\boldsymbol{u}}(u)}{Q_{\boldsymbol{u}}(u)}+Q^-_{\boldsymbol{\theta}}(u)\frac{Q^{++}_{\boldsymbol{u}}(u)}{Q_{\boldsymbol{u}}(u)}\,, \tag{2.8}$$

which is sometimes called the Baxter equation (2.7) for the Q-function defined as

$$Q_{\boldsymbol{u}}(u)=\prod_{i=1}^{M}(u-u_i)\,. \tag{2.9}$$

[7] As in these definitions, each $+$ (respectively $-$) superscript on a function signifies that its argument is shifted by $+\frac{i}{2}$ (respectively $-\frac{i}{2}$). According to this convention, $Q^{++}_{\boldsymbol{\theta}}(u)$ means $Q_{\boldsymbol{\theta}}(u+i)$, etc. When $\theta_i=0$, the functions $Q^{\pm}_{\boldsymbol{\theta}}(u)$ are often referred to as $a(u)$ (for $+$) and $d(u)$ (for $-$).

The equation (2.8) is equivalent to the Bethe ansatz equation through the condition that $t_{\boldsymbol{u}}(u)$ has no poles despite the presence of $Q_{\boldsymbol{u}}(u)$ in the denominator.

The properties of the operators $A(u) \sim D(u)$ under the global SU(2) generators S^i are often quite informative. For instance, from the transformation properties

$$[S^z, B(u)] = -B(u)\,, \tag{2.10}$$

$$[S^+, B(u)] = A(u) - D(u)\,, \tag{2.11}$$

one can easily show that if the Bethe state $|\boldsymbol{u}\rangle$ is on-shell it is the highest weight state with spin $\frac{L}{2} - M$. On the other hand, if it is off-shell, although having the same spin $\frac{L}{2} - M$, it is a direct sum of states belonging to various representations and is not a highest weight state.

2.2 Determinant formulas

The main purpose of this work is to develop a method of computing the scalar product of the form

$$\langle \boldsymbol{v}|\boldsymbol{u}\rangle = \langle \uparrow^L | \prod_{i=1}^{M} C(v_i) \prod_{j=1}^{M} B(u_j) | \uparrow^L \rangle \tag{2.12}$$

for SU(2) spin chain using the SoV formalism, leading to a new integral representation of such a product. Traditionally, however, the computation of such a product has been pursued in the framework of the algebraic Bethe ansatz reviewed in the previous subsection. Although the computation is conceptually quite straightforward as one simply needs to move $C(v_i)$'s all the way through $B(u_j)$'s, using the exchange algebra, and act on the pseudovacuum, in practice this procedure produces a multitude of terms which grow exponentially in the number of magnons and becomes intractable. Fortunately, in the case of the product between an on-shell and an off-shell Bethe states, Slavnov discovered a much more concise expression in the form of a determinant, which was to be called Slavnov's determinant formula [12]. More recently, various other types of determinant formulas have been developed, which are intimately related to the Slavnov's determinant. Since the configuration for which the Slavnov's formula is valid is precisely the one needed for the computation of the three point functions in the super Yang-Mills theory, which motivated our study, it is of interest to sketch in advance that how our new formula will be related, directly or indirectly, to these different variants of determinant formulas.

As stated above, let us consider the case where either one of the set of rapidities $\boldsymbol{u}$ or $\boldsymbol{v}$ are on-shell. Then the original Slavnov's formula computes the scalar product $\langle \boldsymbol{v}|\boldsymbol{u}\rangle$ as

a $M \times M$ determinant of the form

$$\langle \boldsymbol{v}|\boldsymbol{u}\rangle = \frac{\prod_{i=1}^{M} Q^{+}_{\boldsymbol{\theta}}(v_i) Q^{-}_{\boldsymbol{\theta}}(u_i)}{\prod_{i<j}(v_i - v_j)(u_j - u_i)} \times \det\left(\frac{1}{v_i - u_j}\left(\prod_{k\neq j}^{M}(v_i - u_k - i) - \prod_{k\neq j}^{M}(v_i - u_k + i) \prod_{l=1}^{L} \frac{v_i - \theta_l - \frac{i}{2}}{v_i - \theta_i + \frac{i}{2}} \right) \right)_{1\leq i,j\leq M} . \quad (2.13)$$

Very recently, Kostov and Matsuo [22] showed that this expression is equivalent to an alternative determinantal expression of the form

$$\langle \boldsymbol{v}|\boldsymbol{u}\rangle = (-1)^M Z^{\mathrm{KM}}(\boldsymbol{z}|\boldsymbol{\theta}) , \qquad \boldsymbol{z} \equiv \boldsymbol{u} \cup \boldsymbol{v} \quad (2.14)$$

where $Z^{\mathrm{KM}}(\boldsymbol{z}|\boldsymbol{\theta})$ is now a $2M \times 2M$ determinant given by

$$Z^{\mathrm{KM}}(\boldsymbol{z}|\boldsymbol{\theta}) = \frac{\prod_{i=1}^{2M} Q^{-}_{\boldsymbol{\theta}}(z_i)}{\prod_{i<j}(z_i - z_j)} \det\left(z_i^{j-1} - \prod_{l=1}^{L} \frac{z_i - \theta_l + i/2}{z_i - \theta_l - i/2}(z_i + i)^{j-1} \right)_{1\leq i,j\leq 2M} . \quad (2.15)$$

They also pointed out that this equivalence is due essentially to the following equality valid when $\boldsymbol{u}$ or $\boldsymbol{v}$ are on-shell:

$$\langle \uparrow^L | \prod_{i=1}^{M} C(v_i) \prod_{j=1}^{M} B(u_j) | \uparrow^L \rangle \propto \langle \downarrow^L | (S^-)^{L-2M} B(v_i) B(u_i) | \uparrow^L \rangle . \quad (2.16)$$

Intuitively this can be understood in the following way. Suppose the set of rapidities $\boldsymbol{v}$ are on-shell. Then the Bethe state $\prod_{i=1}^{M} B(v_i)|\uparrow^L\rangle$ built on the up vacuum is the highest weight state of global SU(2) with spin $\frac{L}{2} - M$. On the other hand, the state $\prod_{i=1}^{M} C(v_i)|\downarrow^L\rangle$ generated by the action of $C(v)$ on the down pseudovacuum has the same eigenvalue for the transfer matrix $T(u)$. Generally, an on-shell state corresponding to the same solution of the Bethe ansatz equations is expected to belong to the same SU(2) multiplet. Since $\prod_{i=1}^{M} C(v_i)|\downarrow^L\rangle$ is a lowest weight state with spin $-\frac{L}{2} + M$, we can make it into the highest weight state with spin $\frac{L}{2} - M$ by the action of $(S^+)^{L-2M}$. Therefore we should have the equality

$$\prod_{i=1}^{M} B(v_i)|\uparrow^L\rangle \propto (S^+)^{L-2M} \prod_{i=1}^{M} C(v_i)|\downarrow^L\rangle . \quad (2.17)$$

Taking the conjugate of this relation, we obtain (2.16). This identification will be of crucial importance when we develop the SoV method for the computation of the scalar product in section 4.

Now it turns out that our integral formula will be more directly related to another variant of the determinant formula, found by Foda and Wheeler [26]. They showed that

the Kostov-Matsuo expression $Z^{\mathrm{KM}}(\boldsymbol{z})$ can be identified with the so-called partial domain wall partition function (pDWPF) $Z^{\mathrm{pDWPF}}(\boldsymbol{z}|\boldsymbol{\theta})$, which naturally arises in the context of the six vertex model:

$$Z^{\mathrm{pDWPF}}(\boldsymbol{z}|\boldsymbol{\theta}) = \frac{\prod_{\alpha=1}^{2M} Q^{+}_{\boldsymbol{\theta}}(z_\alpha) Q^{-}_{\boldsymbol{\theta}}(z_\alpha)}{\prod_{\alpha<\beta}(z_\alpha - z_\beta)\prod_{i<j}(\theta_j - \theta_i)} \times \det \begin{pmatrix} \frac{i}{(z_1-\theta_1+i/2)(z_1-\theta_1-i/2)} & \cdots & \frac{i}{(z_1-\theta_L+i/2)(z_1-\theta_L-i/2)} \\ \vdots & & \vdots \\ \frac{i}{(z_{2M}-\theta_1+i/2)(z_{2M}-\theta_1-i/2)} & \cdots & \frac{i}{(z_{2M}-\theta_L+i/2)(z_{2M}-\theta_L-i/2)} \\ \theta_1^{L-2M-1} & \cdots & \theta_L^{L-2M-1} \\ \vdots & & \vdots \\ \theta_1^0 & \cdots & \theta_L^0 \end{pmatrix}. \quad (2.18)$$

It is this expression which will be shown, in the Appendix B, to be equivalent to our multiple integral formula. In proving this equality, another more general determinant formula will be of use. It is the Izergin's $L \times L$ determinant [27] expressing the domain wall partition function. A slight generalization of our integral formula will be shown to be equal to this Izergin's large determinant and by taking the limit where $L-2M$ rapidities are sent to infinity and get decoupled, our formula and the Foda-Wheeler formula emerges on the respective side. As the latter can be directly shown to be equivalent to the Slavnov determinant [26], this proves that our formula is an alternative representation of the Slavnov's formula.

3 Separation of variables for integrable models

As reviewed in the previous section, excited states in the XXX spin chain $\prod_i B(u_i)|\uparrow^L\rangle$ are characterized as a collection of magnon excitations on top of the ground state and they are distinguished by a set of complex parameters called the Bethe roots, $\{u_i\}$, which are normally interpreted as *the rapidities of the magnons*. Then the periodicity condition for such excitations leads to the Bethe equation (2.7). In this paper, however, we advocate an alternative view of the excited states, namely that the states are characterized by the nodes (zeros) of their wave functions and the Bethe roots are interpreted instead as *the positions of the nodes*. In this perspective, the Bethe equation arises as a consistency condition for the nodes of the wave function.

To illustrate the basic idea, let us first discuss a simpler example, a one-dimensional harmonic oscillator[8]. As is well-known, the Schrödinger equation for the harmonic os-

[8]This toy model is discussed in a similar manner also in [28]

cillator can be explicitly solved in terms of the Hermite polynomials. However, here we shall take a slightly different route and try to determine the spectrum without explicitly solving the equation. For this purpose, let us first re-express the Schrödinger equation by dividing both sides by the wave function $\psi(x)$:

$$-\frac{\hbar^2}{2m\psi(x)}\frac{d^2}{dx^2}\psi(x)+\frac{m\omega^2x^2}{2}=E\,. \tag{3.1}$$

Then, by studying the behavior of (3.1) at large x, we conclude that $\psi(x)$ should behave as $\psi(x)\sim\exp\left(-m\omega x^2/2\hbar\right)$ when x is large. For excited states, $\psi(x)$ must also contain a polynomial prefactor, which gives rise to nodes of the wave functions. Therefore, to characterize $\psi(x)$ by the position of the nodes, let us write down the following ansatz for $\psi(x)$,

$$\psi(x)=\prod_{i=1}^{N}(x-x_i)e^{-m\omega x^2/2\hbar}\,. \tag{3.2}$$

Substituting this ansatz into (3.1), we obtain the following equation

$$\sum_{i<j}\frac{2}{(x-x_i)(x-x_j)}+\hbar\omega\left(\sum_i\frac{x}{x-x_i}+\frac{1}{2}\right)=E\,. \tag{3.3}$$

Then from its large x behavior, the energy E is determined in terms of the number of nodes as $E=\hbar\omega(N+1/2)$. In addition, since the RHS of (3.3) is a constant and free of poles, we must demand that the residue of the poles at $x=x_i$ on the LHS must vanish. This leads to a Bethe-ansatz-like equation for the positions of the nodes of the wave function,

$$x_i=\frac{\hbar}{2m\omega}\sum_{j\neq i}\frac{1}{x_i-x_j}\,. \tag{3.4}$$

Although this idea of characterizing the excited states in terms of the number and the positions of the nodes is quite elementary and intuitive, it is technically difficult to apply this idea directly to the system with many degrees of freedom. However, in the case of the integrable models, it is often possible to decompose the system into a set of mutually decoupled one dimensional problems. The systematic method to carry this out is the method of separation of variables developed by Sklyanin, which we will explain in the rest of this section. By applying this method, we will see explicitly in section 4 that the Bethe equation for the XXX spin chain can indeed be interpreted as a consistency equation for the nodes of the wave function as in (3.4).

3.1 Basic notions of the separation of variables

Before delving into the details of the method developed by Sklyanin, here we briefly summarize the basic notion of the separation of variables. In classical mechanics, separation of variable is applicable only if there are as many number of conserved charges, $h_1, \ldots, h_d$, as the dynamical variables. In such a case, by a judicious choice of canonical variables, it is often possible to write down a set of equations, each of which contains only one canonical pair $\{x_i, p_i\}$:

$$W_i(x_i, p_i; h_1, \ldots, h_d) = 0\,, \qquad i = 1 \ldots d\,. \tag{3.5}$$

This type of equation is analogous to the expression of the energy of the harmonic oscillator, $E = p^2/2m + m\omega^2 x^2/2$, and one can determine the classical motion of the system in much the same way as in that case.

When we consider the quantum system, the equations (3.5) are replaced by the following equations for the eigenstates of the conserved charges,

$$W_i(\hat{x}_i, \hat{p}_i; h_1, \ldots, h_d)|\Psi\rangle = 0\,, \qquad i = 1 \ldots d\,. \tag{3.6}$$

In terms of the wave function in the coordinate representation, $\Psi(x_1, \ldots, x_d)$, (3.6) can be re-expressed as

$$W_i\left(x_i, \frac{\hbar}{i}\frac{\partial}{\partial x_i}; h_1, \ldots, h_d\right)\Psi(x_1, \ldots, x_d) = 0\,, \qquad i = 1 \ldots d\,. \tag{3.7}$$

It is easy to see that (3.7) admits a completely factorized solution, $\Psi = \prod_i \psi_i(x_i)$, each factor of which satisfies the following one dimensional equation,

$$W_i\left(x_i, \frac{\hbar}{i}\frac{\partial}{\partial x_i}; h_1, \ldots, h_d\right)\psi_i(x_i) = 0\,. \tag{3.8}$$

In this way, the original system with many degrees of freedom can be reduced to a set of mutually decoupled one dimensional systems.

3.2 Sklyanin's magic recipe

The most nontrivial step in the procedure above is the construction of the separated variables satisfying the equations of the form (3.5) or (3.6). This is indeed a difficult problem for interacting many-body systems. However, for the integrable models which can be formulated in terms of the Lax operators, Sklyanin proposed a systematic method for the construction, often referred to as the Sklyanin's magic recipe. In what follows, we sketch the essence of this recipe[9] applied to systems with a 2×2 monodromy matrix.

[9]The discussion here is basically restricted to the simplest class of the integrable models, called *rational* models. For *trigonometric* or *elliptic* models, nontrivial modification of the method is required [18].

More precise analysis for the case of the XXX spin chain will be given in the next section.

For simplicity, let us first consider the classical case. In a classically integrable system with a 2×2 monodromy matrix

$$\Omega(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} , \tag{3.9}$$

there is an immediate candidate for the set of equations (3.5). It is the *characteristic equation* for the monodromy matrix

$$\det\left(z - \Omega(x)\right) = 0 , \tag{3.10}$$

where z is the eigenvalue of the matrix $\Omega(x)$. Since the expansion of $\Omega(x)$ in powers of x yields a set of conserved charges as its coefficients, (3.10) is indeed of the form of (3.5) if we can somehow identify x and z with dynamical variables. The recipe proposed by Sklyanin is to use the solutions x_i's to the equation $B(u) = 0$ as x-variables:

$$B(u) = (u - x_1)(u - x_2) \cdots . \tag{3.11}$$

In the case of the lattice models, such as the XXX spin chain discussed in the previous section, $B(u)$ is a polynomial in u, whose order basically equals the lattice size. Therefore, this prescription indeed provides the correct number of variables. Furthermore, owing to the Poisson commutativity among $B(u)$'s, x_i's also commute with each other and thus they are mutually independent separated variables. On the other hand, the z-variables, which are the eigenvalues of $\Omega(x)$, are provided by the diagonal components, $A(x_i)$ or $D(x_i)$, since $\Omega(x_i)$ becomes a lower triangular matrix owing to $B(x_i) = 0$. Then the remaining task is to understand the relation of $A(x_i)$ and $D(x_i)$ to the conjugate momenta p_i, which satisfy the standard commutation relations:

$$\{x_i, x_j\} = 0 , \qquad \{p_i, p_j\} = 0 , \qquad \{x_i, p_j\} = \delta_{ij} . \tag{3.12}$$

In most cases, by explicitly computing the Poisson brackets of $A(x_i)$ and $D(x_i)$ with x_i, we can show that they are related to p_i roughly as

$$A(x_i) \sim e^{ip_i} , \qquad D(x_i) \sim e^{-ip_i} . \tag{3.13}$$

In the case of the quantum integrable models, separated variables x_i's become a set of commuting operators $\hat{x}_i$'s, which are characterized as the roots of the *operator* equation $B(u) = 0$. Just as for the classical case, the conjugate operators, $e^{i\hat{p}_i}$ and $e^{-i\hat{p}_i}$, are given[10]

[10]Note, in the quantum case, we need to consider the ordering of the operators. In the case of the XXX spin chain, this is explicitly worked out in section 3.

essentially by $A(\hat{x}_i)$ or $D(\hat{x}_i)$. To derive a set of one dimensional equations of the type (3.7), let us consider the wave function in the x_i-basis:

$$\Psi(x_1, \ldots, x_d) = \langle x_1, \ldots, x_d | \Psi \rangle \,, \tag{3.14}$$

where $\langle x_1, \ldots, x_d|$ is an eigenstate of the operators $\hat{x}_i$'s. Now if the state $|\Psi\rangle$ is an eigenstate of $T(u) = A(u) + D(u)$, a generating function of the commuting set of Hamiltonians, we can compute $\langle x_1, \ldots, x_d | T(\hat{x}_i) | \Psi \rangle$ as

$$\langle x_1, \ldots, x_d | T(\hat{x}_i) | \Psi \rangle = t(x_i) \Psi(x_1, \ldots, x_d) \,, \tag{3.15}$$

where $t(u)$ is the eigenvalue of $T(u)$ for $|\Psi\rangle$. We can evaluate the same quantity also by acting $T(\hat{x}_i)$ to the left on $\langle x_1, \ldots, x_d|$. To carry this out we use the relation of $T(\hat{x}_i)$ with the momenta $\hat{p}_i$, *i.e.*

$$T(\hat{x}_i) = A(\hat{x}_i) + D(\hat{x}_i) \sim e^{i\hat{p}_i} + e^{-i\hat{p}_i} \,. \tag{3.16}$$

Then we find

$$\langle \ldots, x_i, \ldots | T(\hat{x}_i) \sim \langle \ldots, x_i + 1, \ldots | + \langle \ldots, x_i - 1, \ldots | \,. \tag{3.17}$$

In this way we arrive at the following equation for the wave function Ψ:

$$t(x_i) \Psi(\ldots, x_i, \ldots) \sim \Psi(\ldots, x_i + 1, \ldots) + \Psi(\ldots, x_i - 1, \ldots) \,, \quad i = 1, \ldots, d \,. \tag{3.18}$$

Assuming the factorized form of the wave functions $\Psi(x_1, \ldots, x_d) = \prod_i \psi(x_i)$, we can decompose (3.18) into a set of mutually decoupled one dimensional equations:

$$t(x_i) \psi_i(x_i) \sim \psi(x_i + 1) + \psi(x_i - 1) \,. \tag{3.19}$$

This equation is the analogue of the Schrödinger equation for the harmonic oscillator. Therefore, as in that case, we can derive a consistency condition for the nodes of the wave function. Assuming a form of ψ_i as $\psi_i(x) = \prod_k (x - u_k)$ and setting $x_i = u_k$ in (3.19), we obtain the algebraic relations for the positions u_k of the nodes:

$$1 \sim \prod_{j \neq k} \frac{u_k - u_j + 1}{u_k - u_j - 1} \,. \tag{3.20}$$

Note that this is identical with the Bethe equation. Therefore, as mentioned at the beginning of this section, the Bethe roots can be interpreted as the nodes of the wave function in this approach. In the next section, we will see that the logic outlined here is explicitly realized in the case of the XXX spin chain.

4 Integral representation of the scalar products for XXX spin chain

In the preceding section, we sketched the basic idea of the method of separated variables for integrable models. In this section, we will apply it to the periodic SU(2) XXX spin chain and derive a multiple integral representation of the scalar products in the basis where the separated variables are diagonal. The resultant expression can be brought to a form which resembles the integral over the eigenvalues of a matrix model.

4.1 Construction of the separated variables

Recall the definition of the monodromy matrix $\Omega(u)$ for the XXX spin chain with inhomogeneity parameters θ_i:

$$\Omega(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \equiv L_1(u-\theta_1)L_2(u-\theta_2)\cdots L_L(u-\theta_L)\,, \tag{4.1}$$

$$L_i(u) \equiv \begin{pmatrix} u+iS_i^z & iS_i^- \\ iS_i^+ & u-iS_i^z \end{pmatrix}. \tag{4.2}$$

As outlined in the previous section, the separated variables for integrable models with a 2×2 monodromy matrix are usually given by the roots of the operator equation, $B(u)=0$. However, as already pointed out in the introduction, in the case of the periodic SU(2) spin chain, the operator $B(u)$ is proportional to S^- in the large u limit as $B(u)\sim iS^- u^{L-1}+\cdots$, and is not diagonalizable. This problem can be circumvented by introducing a twisting matrix $K_\epsilon = \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix}$, which changes the boundary condition and modifies the monodromy matrix as

$$\Omega_\epsilon(u) = K_\epsilon \Omega(u) \equiv \begin{pmatrix} A_\epsilon(u) & B_\epsilon(u) \\ C_\epsilon(u) & D_\epsilon(u) \end{pmatrix}. \tag{4.3}$$

Although such a twisting changes the dynamical properties of the spin chain, it does not affect the computation of the scalar products since, as we shall show explicitly later in this section, they can be re-expressed in terms of quantities independent of the twisting parameter ϵ. After twisting, the large u behavior of $B_\epsilon(u)$ is modified to $B_\epsilon(u)\sim \epsilon u^L + i(S^- - \epsilon S^z + i\sum_j \theta_j)u^{L-1}+\cdots$ and $B_\epsilon(u)$ becomes diagonalizable. Then it can be factorized as

$$B_\epsilon(u) = \epsilon \prod_{i=1}^{L}(u-\hat{x}_i)\,, \tag{4.4}$$

where $\hat{x}_i$'s are the roots of the operator equation, $B_\epsilon(u) = 0$. As the twisting preserves the algebra among the elements $A(u) \sim D(u)$, the operators B_ϵ's continue to commute with each other, namely $[B_\epsilon(u), B_\epsilon(v)] = 0$, and this implies that $\hat{x}_i$'s also mutually commute: $[\hat{x}_i\,, \hat{x}_j] = 0$. These operators are the "coordinates" of the separated variables and one can consider their left eigenstates and right eigenstates, $\langle x_1, \ldots, x_L|$ and $|x_1, \ldots, x_L\rangle$, upon which $B_\epsilon(u)$ acts in the following way:

$$\langle x_1, \ldots, x_L | B_\epsilon(u) = \left(\epsilon \prod_{i=1}^{L} (u - x_i) \right) \langle x_1, \ldots, x_L | \,, \tag{4.5}$$

$$B_\epsilon(u) | x_1, \ldots, x_L \rangle = \left(\epsilon \prod_{i=1}^{L} (u - x_i) \right) | x_1, \ldots, x_L \rangle \,. \tag{4.6}$$

As explained in the Appendix A, the eigenvalue of the operator $\hat{x}_i$ takes only two values given by $\hat{x}_i \in \left\{\theta_i - \frac{i}{2}\,, \theta_i + \frac{i}{2}\right\}$. As a consequence, the dimension of the Hilbert space spanned by the eigenstates of the separated variables is 2^L, which precisely matches that of the spin chain Hilbert space. This assures the completeness of the separated variable basis.

At $\hat{x}_i$ the operator B_ϵ vanishes and the form of the monodromy matrix becomes lower triangular. Therefore the two eigenvalues are given by $A_\epsilon(\hat{x}_i)$ and $D_\epsilon(\hat{x}_i)$, which are expected to be identified as $e^{\pm i\hat{p}_i}$, where $\hat{p}_i$ is the momentum operator conjugate to $\hat{x}_i$. To see this more precisely, since $A_\epsilon(u)$ and $D_\epsilon(u)$ are polynomials in u with operator-valued coefficients, we need to specify the ordering of $\hat{x}_i$ and the coefficients, which are also operators in general. The ordering appropriate for the left eigenstates, to be denoted by $:***:_L$, turns out to be placing all the $\hat{x}_i$'s to the left of the coefficients, namely

$$:F(\hat{x}_i):_L \equiv \sum_n \hat{x}_i^n \hat{F}_n \,, \qquad \text{for} \quad F(u) = \sum_n u^n \hat{F}_n \,. \tag{4.7}$$

Then the commutation relation between $A_\epsilon(u)$ and $B_\epsilon(u)$, given by $(u-v)A_\epsilon(v)B_\epsilon(u) = (u-v+i)B_\epsilon(u)A_\epsilon(v) - iB_\epsilon(v)A_\epsilon(u)$, leads to

$$:(u - \hat{x}_i)A_\epsilon(\hat{x}_i)B_\epsilon(u):_L \,= \,:(u - \hat{x}_i + i)B_\epsilon(u)A_\epsilon(\hat{x}_i):_L - i:B_\epsilon(\hat{x}_i)A_\epsilon(u):_L \,. \tag{4.8}$$

Since the second term on the RHS of (4.8), containing $B_\epsilon(\hat{x}_i)$, vanishes, and since $B_\epsilon(u)$ commutes with $\hat{x}_i$, we can simplify (4.8) to

$$(u - \hat{x}_i):A_\epsilon(\hat{x}_i):_L B_\epsilon(u) = (u - \hat{x}_i + i)B_\epsilon(u):A_\epsilon(\hat{x}_i):_L \,, \tag{4.9}$$

where the normal-ordering is now imposed only on $A_\epsilon(\hat{x}_i)$. Then by acting (4.9) to the left eigenstate, we obtain

$$(u - x_i)\langle x_1, \ldots, x_L|:A_\epsilon(\hat{x}_i):_L B_\epsilon(u) = \epsilon(u - x_i + i) \prod_{k=1}^{L} (u - x_k) \langle x_1, \ldots, x_L|:A_\epsilon(\hat{x}_i):_L \,.$$

Dividing both sides by $(u-x_i)$, we see that $B(u)$ acting on the state $\langle x_1,\ldots,x_L|{:}A_\epsilon(\hat{x}_i){:}_L$ vanishes at $u=x_i-i$. This means that the operator ${:}A_\epsilon(\hat{x}_i){:}_L$ indeed effects the shift of the eigenvalue of $\hat{x}_i$ by $-i$, namely[11]

$$\langle\ldots,x_i,\ldots|{:}A_\epsilon(\hat{x}_i){:}_L \propto \langle\ldots,x_i-i,\ldots|\,. \tag{4.10}$$

A similar argument for $D_\epsilon(u)$ leads to the conclusion that ${:}D_\epsilon(\hat{x}_i){:}_L$ shifts the eigenvalue of $\hat{x}_i$ by $+i$,

$$\langle\ldots,x_i,\ldots|{:}D_\epsilon(\hat{x}_i){:}_L \propto \langle\ldots,x_i+i,\ldots|\,. \tag{4.11}$$

The constants of proportionality in (4.10) and (4.11) can be determined by the analysis detailed in the Appendix A. Since these results, together with the spectrum of $\hat{x}_i$ already quoted, are basic to the rest of the analysis, we shall display them as a theorem:

Theorem 1:

(i) The spectrum of $\hat{x}_i$ is given by

$$\hat{x}_i \in \left\{\theta_i-\frac{i}{2}\,,\theta_i+\frac{i}{2}\right\}\,. \tag{4.12}$$

(ii) The operators ${:}A_\epsilon(\hat{x}_i){:}_L$ and ${:}D_\epsilon(\hat{x}_i){:}_L$ act on the left eigenstates in the following manner

$$\langle\ldots,x_i,\ldots|{:}A_\epsilon(\hat{x}_i){:}_L = \sqrt{1+\epsilon^2}\,Q^+_{\boldsymbol{\theta}}(x_i)\langle\ldots,x_i-i,\ldots|\,, \tag{4.13}$$

$$\langle\ldots,x_i,\ldots|{:}D_\epsilon(\hat{x}_i){:}_L = \sqrt{1+\epsilon^2}\,Q^-_{\boldsymbol{\theta}}(x_i)\langle\ldots,x_i+i,\ldots|\,. \tag{4.14}$$

For the right eigenstates, an appropriate ordering prescription is to put all x_i's to the right of the coefficients of $A_\epsilon(u)$ and $D_\epsilon(u)$:

$${:}F(\hat{x}_i){:}_R \equiv \sum_n \hat{F}_n\hat{x}_i^n\,, \qquad\qquad \text{for}\quad F(u)=\sum_n u^n\hat{F}_n\,. \tag{4.15}$$

Then the action of ${:}A_\epsilon(\hat{x}_i){:}_R$ and ${:}D_\epsilon(\hat{x}_i){:}_R$ on the right eigenstates are expressible as

$${:}A_\epsilon(\hat{x}_i){:}_R|\ldots,x_i,\ldots\rangle = \sqrt{1+\epsilon^2}\,Q^-_{\boldsymbol{\theta}}(x_i)|\ldots,x_i+i,\ldots\rangle\,, \tag{4.16}$$

$${:}D_\epsilon(\hat{x}_i){:}_R|\ldots,x_i,\ldots\rangle = \sqrt{1+\epsilon^2}\,Q^+_{\boldsymbol{\theta}}(x_i)|\ldots,x_i-i,\ldots\rangle\,. \tag{4.17}$$

Since $A_\epsilon(u)$ and $D_\epsilon(u)$ are L-th order polynomials in u with a unit leading coefficient, the action of these operators at L distinct values of u, (4.13) and (4.14), completely determines

[11] For the literal identification of ${:}A_\epsilon(\hat{x}_i){:}_L$ with $e^{i\hat{p}_i}$, it is more natural to rename $\hat{x}_i$ as $-i\hat{x}_i$. Then, the new $\hat{x}_i$ gets shifted by $+1$ and its spectrum becomes real at $\theta_i=0$. But we shall not do this and stick to the customary definition.

the explicit forms of the operators as follows:

$$A_\epsilon(u) = \prod_{k=1}^{L}(u-\hat{x}_k) + \sum_{k=1}^{L}\left(\prod_{j\neq k}\frac{u-\hat{x}_j}{\hat{x}_k-\hat{x}_j}\right) :A_\epsilon(\hat{x}_k):_L\,, \tag{4.18}$$

$$D_\epsilon(u) = \prod_{k=1}^{L}(u-\hat{x}_k) + \sum_{k=1}^{L}\left(\prod_{j\neq k}\frac{u-\hat{x}_j}{\hat{x}_k-\hat{x}_j}\right) :D_\epsilon(\hat{x}_k):_L\,. \tag{4.19}$$

They are expressible also in terms of the right-ordered operators, $:A_\epsilon(\hat{x}_i):_R$ and $:D_\epsilon(\hat{x}_i):_R$, as

$$A_\epsilon(u) = \prod_{k=1}^{L}(u-\hat{x}_k) + \sum_{k=1}^{L}:A_\epsilon(\hat{x}_k):_R\left(\prod_{j\neq k}\frac{u-\hat{x}_j}{\hat{x}_k-\hat{x}_j}\right)\,, \tag{4.20}$$

$$D_\epsilon(u) = \prod_{k=1}^{L}(u-\hat{x}_k) + \sum_{k=1}^{L}:D_\epsilon(\hat{x}_k):_R\left(\prod_{j\neq k}\frac{u-\hat{x}_j}{\hat{x}_k-\hat{x}_j}\right)\,. \tag{4.21}$$

From (4.18) and (4.19), we can derive a difference equation for the eigenstate $|\psi\rangle$ of the (twisted) transfer matrix, $T_\epsilon(u) \equiv A_\epsilon(u) + D_\epsilon(u)$. This is done by computing $\langle x_1, x_2, \ldots, x_L|T_\epsilon(u)|\psi\rangle$ in two different ways: First by acting $T_\epsilon(u)$ on $\langle x_1, x_2, \ldots, x_L|$ using (4.18) and (4.19), and second by acting it on $|\psi\rangle$. By setting $u = x_i$ in the resulting equation, we obtain the following simple equation for the wave function of the eigenstate, $\Psi(x_1, \ldots, x_L) \equiv \langle x_1, \ldots, x_L|\psi\rangle$:

$$\frac{t_\epsilon(x_i)}{\sqrt{1+\epsilon^2}}\Psi(\ldots, x_i, \ldots) = Q^+_{\boldsymbol{\theta}}(x_i)\Psi(\ldots, x_i - i, \ldots) + Q^-_{\boldsymbol{\theta}}(x_i)\Psi(\ldots, x_i + i, \ldots)\,. \tag{4.22}$$

Here $t_\epsilon(u)$ is the eigenvalue of $T_\epsilon(u)$, *i.e.* $T_\epsilon(u)|\psi\rangle = t_\epsilon(u)|\psi\rangle$. Assuming a factorized form of the wave function, $\Psi(x_1, x_2, \ldots, x_L) = \psi_1(x_1)\psi_2(x_2)\ldots\psi_L(x_L)$, (4.22) can be decomposed into a set of L one-dimensional equations, which can be regarded as the "Schrödinger equations" for the separated variables:

$$\frac{t_\epsilon(x_i)}{\sqrt{1+\epsilon^2}}\psi_i(x_i) = Q^+_{\boldsymbol{\theta}}(x_i)\psi_i^{--}(x_i) + Q^-_{\boldsymbol{\theta}}(x_i)\psi_i^{++}(x_i)\,. \tag{4.23}$$

In the $\epsilon \to 0$ limit, the equation (4.23) for ψ_i apparently takes the same form as the Baxter equation (2.8) for the Q-function, $Q_{\boldsymbol{u}}(u)$. However one should keep in mind that $Q_{\boldsymbol{u}}$ and $\psi_i(x_i)$ are conceptually quite different: While $Q_{\boldsymbol{u}}$ is introduced as a polynomial with zeros at the rapidities of the magnon excitations and can be defined on the whole complex plane, ψ_i is the wave function in the separated variable basis and is defined only on the discrete eigenvalues $x_i = \theta_i \pm i/2$. Therefore it is *a priori* not clear whether we can identify ψ_i with $Q_{\boldsymbol{u}}$. Nevertheless, as we shall later see explicitly, the factor representing

the wave function in the multiple integral formula is given indeed by the Q-function. Therefore, as far as the multiple integral formula is concerned, ψ_i can be identified with $Q_{\boldsymbol{u}}$ and the Bethe equation can be interpreted as the consistency condition for the zeros of the wave function. This evidently parallels the case of the harmonic oscillator d iscussed in section 3.

4.2 Multiple integral representation for scalar products

Having constructed the separated variables, our next goal is to express the scalar product between an off-shell and an on-shell Bethe states given by $\langle \boldsymbol{v}|\boldsymbol{u}\rangle = \langle\uparrow^L | \prod_{i=1}^M C(v_i) \prod_{i=1}^M B(u_i)| \uparrow^L\rangle$ as the overlap between two wave functions of separated variables. Our basic strategy for deriving such a expression is to insert into the scalar product a resolution of unity in the SoV basis, namely

$$\mathbf{1} = \sum_{x_i=\theta\pm i/2} \mu(\boldsymbol{x})\, |\boldsymbol{x}\rangle\langle\boldsymbol{x}|\,, \tag{4.24}$$

where $\boldsymbol{x}$ stands for $\{x_1,\dots,x_L\}$ and $\mu(\boldsymbol{x})$ is the measure factor for the summation, to be specified later. Unfortunately, this procedure cannot be carried out straightforwardly because the scalar product of our interest contains the operator $C(u)$ and its action on the B-diagonal SoV basis is quite complicated. In addition, to employ the SoV basis, we need to introduce the twist in the boundary condition as in (4.3), which is not present in the original scalar product as above.

The first problem can be circumvented by the trick due to Kostov and Matsuo [22], which converts $C(v_i)$ to $B(v_i)$ within the scalar product provided v_i's satisfy the Bethe equation. Although not explicitly given in [22], one can work out the precise factors in the conversion formula and obtain the expression

$$\langle\uparrow^L | \prod_{i=1}^M C(v_i) \prod_{j=1}^M B(u_j)| \uparrow^L\rangle = \frac{(-1)^M}{(L-2M)!} \langle\downarrow^L |(S^-)^{L-2M} \prod_{i=1}^M B(v_i) \prod_{j=1}^M B(u_j)| \uparrow^L\rangle\,, \tag{4.25}$$

which contains only the operator $B(u)$. This rewriting has another gratifying feature: It allows us to introduce the twist of the boundary condition without changing the value of the scalar product. This is done by replacing the second line in (4.25) with

$$\langle\downarrow^L |(S^- - \epsilon S^z + i\sum_{j=1}^L \theta_j)^{L-2M} \prod_{i=1}^M B_\epsilon(v_i) \prod_{j=1}^M B_\epsilon(u_j)| \uparrow^L\rangle\,. \tag{4.26}$$

Although (4.26) has apparent dependence on ϵ as well as an extra dependence on θ_i's,

such unwanted terms actually vanish[12] thanks to the conservation of the total spin S^z along the z-axis.

Then, inserting a resolution of unity (4.24) into (4.26) and using the action of $B_\epsilon(u)$ on the SoV basis (4.5) and (4.6), we obtain the following expression[13]

$$\begin{aligned}&\langle\downarrow^L|(S^- - \epsilon S^z + i\sum_{j=1}^{L}\theta_j)^{L-2M}\prod_{i=1}^{M}B_\epsilon(v_i)\prod_{j=1}^{M}B_\epsilon(u_j)|\uparrow^L\rangle\\ &= \sum_{x_i=\theta\pm i/2}\epsilon^L\mu(\boldsymbol{x})f_L(\boldsymbol{x})f_R(\boldsymbol{x})\left(\sum_{j=1}^{L}x_j\right)^{L-2M}\prod_{k=1}^{L}Q_{\boldsymbol{u}}(x_k)Q_{\boldsymbol{v}}(x_k)\,,\end{aligned}\tag{4.27}$$

where $f_{L,R}$ are given by

$$f_L(\boldsymbol{x})\equiv\langle\downarrow^L|\boldsymbol{x}\rangle\,,\qquad f_R(\boldsymbol{x})\equiv\langle\boldsymbol{x}|\uparrow^L\rangle\,.\tag{4.28}$$

Note that both the measure $\mu(\boldsymbol{x})$ and the functions $f_{L,R}(\boldsymbol{x})$ depend on the twist parameter ϵ but the total expression (4.27) should be ϵ-independent as argued above.

Let us now determine $\mu(\boldsymbol{x})$ and $f_{L,R}(\boldsymbol{x})$. Note that the measure μ appears in the norm between the left and the right eigenstates of in the SoV basis as

$$\langle\boldsymbol{x'}|\boldsymbol{x}\rangle=\mu^{-1}(\boldsymbol{x})\,\delta_{x'_1x_1}\dots\delta_{x'_Lx_L}\,.\tag{4.29}$$

This suggests that $\mu(\boldsymbol{x})$ can be determined by computing the matrix element

$$\langle\boldsymbol{x'}|A_\epsilon(u)|\boldsymbol{x}\rangle\tag{4.30}$$

in two different ways: First by acting $A_\epsilon(u)$ on the bra using (4.13) and (4.18) and second by acting $D_\epsilon(u)$ on the ket using (4.16) and (4.20). By equating these two expressions, we can easily arrive at the following recursion relation for $\mu(\boldsymbol{x})$:

$$\frac{\mu(\dots,x_k+i,\dots)}{\mu(\dots,x_k,\dots)}=\frac{Q^-_{\boldsymbol{\theta}}(x_k)}{Q^{+++}_{\boldsymbol{\theta}}(x_k)}\prod_{j\neq k}\frac{x_k-x_j+i}{x_k-x_j}\,.\tag{4.31}$$

The solution to this equation can be easily obtained as

$$\mu(\boldsymbol{x})\propto\prod_{i<j}(x_i-x_j)\prod_k e^{\pi(x_k-\theta_k)}\prod_{l\neq m}\frac{1}{(x_l-\theta_m)^2+1/4}\,.\tag{4.32}$$

[12] To see this, it suffices to recall that $B_\epsilon(u)$ is composed of $B(u)+\epsilon D(u)$ and S^- and that $B(u)$ lowers the eigenvalue of S^z by $1/2$ while S^z and $D(u)$ leave it unchanged.

[13] Note that the combination $S^- - \epsilon S^z + i\sum_j\theta_j$ appears in $B_\epsilon(u)$ as $B_\epsilon(u)\sim\epsilon u^L + i(S^- - \epsilon S^3 + i\sum_j\theta_j)u^{L-1}+\dots$ and its action on the SoV basis is thus given by $(S^- - \epsilon S^3 + i\sum_j\theta_j)|x_1,\dots,x_L\rangle = \epsilon\sum_i x_i|x_1,\dots,x_L\rangle$.

Similarly, by computing $\langle\downarrow^L|A_\epsilon(u)|\boldsymbol{x}\rangle$ and $\langle\boldsymbol{x}|A_\epsilon(u)|\uparrow^L\rangle$ using the formulas

$$\langle\downarrow^L|A_\epsilon(u) = \langle\downarrow^L|\,(A(u)+\epsilon C(u)) = Q^-_{\boldsymbol{\theta}}(u)\langle\downarrow^L|\,, \tag{4.33}$$

$$A_\epsilon(u)|\uparrow^L\rangle = (A(u)+\epsilon C(u))\,|\uparrow^L\rangle = Q^+_{\boldsymbol{\theta}}(u)|\uparrow^L\rangle\,, \tag{4.34}$$

we can derive the recursion relations for $f_{L,R}(\boldsymbol{x})$. Solving those recursion relations, $f_{L,R}(\boldsymbol{x})$ can be determined as

$$f_L(\boldsymbol{x}) \propto \exp\left(-\frac{i}{2}\ln(1+\epsilon^2)\sum_{k=1}^{L}x_k\right)\,,\quad f_R(\boldsymbol{x}) \propto \exp\left(\frac{i}{2}\ln(1+\epsilon^2)\sum_{k=1}^{L}x_k\right)\,. \tag{4.35}$$

The constants of proportionality in (4.32) and (4.35), which are left undetermined, are functions of the twist parameter ϵ and the inhomogeneity parameters θ_i's. These constants are related to the overall normalization of the scalar products and will be fixed by the analysis presented in the Appendix B, which compares it with the other known formula for the scalar product. Let us now convert the summation over the discrete spectrum of $\hat{x}_i$'s to contour integrals over the continuous variables x_i. This can be done by utilizing the pole structure in (4.32), namely

$$\frac{1}{(x_l-\theta_m)^2+1/4} = \frac{1}{(x_l-\theta_m+\frac{i}{2})(x_l-\theta_m-\frac{i}{2})} \tag{4.36}$$

Therefore, if we recall the definition of the function $Q_{\boldsymbol{\theta}}$ given in (2.6), we can write the measure as the residue

$$\mu(\boldsymbol{x}) = \mathrm{Res}_{z_i=x_i}\left[\frac{\prod_{i<j}(z_i-z_j)}{\prod_k Q^+_{\boldsymbol{\theta}}(z_k)Q^-_{\boldsymbol{\theta}}(z_k)}\right]\,, \tag{4.37}$$

and perform the conversion into contour integrals. Taking into account the constants of proportionality in (4.32) and (4.35), we finally arrive at the following multiple integral formula[14] for the scalar product between an off-shell Bethe-state and an on-shell Bethe state:

$$\begin{aligned}\langle\uparrow^L|\prod_{i=1}^{M}C(v_i)\prod_{j=1}^{M}B(u_j)|\uparrow^L\rangle = \frac{\prod_{i<j}(\theta_i-\theta_j)(\theta_i-\theta_j+i)(\theta_i-\theta_j-i)}{(L-2M)!}\\ \times\prod_{i=1}^{L}\oint_{\mathcal{C}_i}\frac{dx_i}{2\pi i}\left(\sum_{j=1}^{L}x_j\right)^{L-2M}\prod_{k<l}(x_k-x_l)\prod_{m=1}^{L}\frac{Q_{\boldsymbol{u}}(x_m)Q_{\boldsymbol{v}}(x_m)}{Q^+_{\boldsymbol{\theta}}(x_m)Q^-_{\boldsymbol{\theta}}(x_m)}\,.\end{aligned} \tag{4.38}$$

In this formula, $\mathcal{C}_i$ denotes the integration contour which encloses $\theta_i\pm i/2$ counterclockwise. Actually the prefactors in front of the integral are unimportant when computing physical observables, since they drop out upon normalizing the Bethe states.

[14] In the Appendix B, we will give a direct analytical proof of the equivalence between (4.38) and the known determinant formulas.

4.3 Symmetrization and simplification of the multiple integral

The multiple integral formula (4.38) derived in the last subsection has one unsatisfactory feature: This expression becomes singular as we take the homogeneous limit, $\theta_i \to 0$. There are two sources for the singular behavior. One is that the integration contours $\mathcal{C}_i$ get pinched and collide when all the θ_i's move to the origin. Another source is that at the same time the prefactor $\prod_{i<j}(\theta_i - \theta_j)$ will vanish. To get around this difficulty, we wish to deform each integration contour into the one, to be denoted by $\mathcal{C}_{\rm all}$, which encloses all the singularities in the integrand. However, if we naïvely make such deformations, obviously we will pick up unwanted contributions coming from different integration variables encircling the poles from the same group $\theta_i \pm i/2$. We can avoid such contributions by inserting a factor of the form $\prod_{k<l}(e^{2\pi x_k} - e^{2\pi x_l})$, which vanishes for all the undesired cases. For the genuine contributions for which this factor does not vanish, we must normalize properly to reproduce the original value of the integral. In this way, with the factor $L!$ coming from the permutation of x_i's, we obtain the following more symmetric expression for the scalar product:

$$\langle\uparrow^L | \prod_{i=1}^{M} C(v_i) \prod_{j=1}^{M} B(u_j) | \uparrow^L\rangle = \frac{\Xi}{L!(L-2M)!} \times \prod_{i=1}^{L} \oint_{\mathcal{C}_{\rm all}} \frac{dx_i}{2\pi i} \left(\sum_{j=1}^{L} x_j\right)^{L-2M} \prod_{k<l}(x_k - x_l)(e^{2\pi x_k} - e^{2\pi x_l}) \prod_{m=1}^{L} \frac{Q_{\boldsymbol{u}}(x_m) Q_{\boldsymbol{v}}(x_m)}{Q^+_{\boldsymbol{\theta}}(x_m) Q^-_{\boldsymbol{\theta}}(x_m)}\,, \quad (4.39)$$

where the prefactor Ξ is given by

$$\Xi \equiv \prod_{i<j} \frac{(\theta_i - \theta_j)(\theta_i - \theta_j + i)(\theta_i - \theta_j - i)}{(e^{2\pi\theta_j} - e^{2\pi\theta_i})}\,. \quad (4.40)$$

Note that for this expression the prefactor Ξ is indeed finite in the homogeneous limit.

Although the expression above is symmetric in all the variables and hence quite useful, it is of interest to point out that actually we can integrate out one of the x_i's and obtain a slightly simpler expression containing $L-1$ integration variables. To derive it, let us first re-express the factor $\prod_{k<l}(e^{2\pi x_k} - e^{2\pi x_l})$ as a determinant of Vandermonde type:

$$\prod_{k<l}(e^{2\pi x_k} - e^{2\pi x_l}) = \det\left(e^{2\pi(j-1)x_k}\right)_{1\le j,k\le L}\,. \quad (4.41)$$

Then, by using the basic definition of the determinant, we can rewrite it into a sum over permutations of the form $\sum_\sigma (-1)^\sigma e^{2\pi(\sigma(j)-1)x_j}$. Now note that the remaining terms in the integrand is completely antisymmetric with respect to permutations. Hence, all the

terms in the above sum contribute equally and we arrive at the following expression:

$$\frac{\Xi}{(L-2M)!}\prod_{i=1}^{L}\oint_{\mathcal{C}_{\rm all}}\frac{dx_i}{2\pi i}\left(\sum_{j=1}^{L}x_j\right)^{L-2M}\prod_{k<l}(x_k-x_l)\prod_{m=1}^{L}\frac{Q_{\boldsymbol{u}}(x_m)Q_{\boldsymbol{v}}(x_m)e^{2\pi(m-1)x_m}}{Q^+_{\boldsymbol{\theta}}(x_m)Q^-_{\boldsymbol{\theta}}(x_m)}\,. \quad (4.42)$$

Notice that the integral is over meromorphic factors, except for $\exp(2\pi(m-1)x_m)$. However for x_1 this factor is absent. Hence we can easily integrate out this variable by closing its contour at infinity. At infinity all the factors become power functions and the only non-vanishing integral to be performed is $\oint dx_1/(2\pi i x_1)=1$. After this procedure, we may convert the factor $e^{2\pi(m-1)x_m}$ back to the determinant and further to the original expression $\prod_{k<l}(e^{2\pi x_k}-e^{2\pi x_l})$. In this way we obtain the following simple formula with $L-1$ integration variables:

$$\begin{aligned}\langle\uparrow^L|\prod_{i=1}^{M}C(v_i)\prod_{j=1}^{M}B(u_j)|\uparrow^L\rangle &= \frac{\Xi}{(L-1)!(L-2M)!}\\ &\times\prod_{i=1}^{L-1}\oint_{\mathcal{C}_{\rm all}}\frac{dx_i}{2\pi i}\prod_{k<l}(x_k-x_l)(e^{2\pi x_k}-e^{2\pi x_l})\prod_{m=1}^{L-1}\frac{Q_{\boldsymbol{u}}(x_m)Q_{\boldsymbol{v}}(x_m)e^{2\pi x_m}}{Q^+_{\boldsymbol{\theta}}(x_m)Q^-_{\boldsymbol{\theta}}(x_m)}\,. \quad (4.43)\end{aligned}$$

Note that the factor, $(\sum_i x_i)^{L-2M}$, which was present in the previous expressions, disappeared upon integration over x_1. Therefore (4.43) is structurally similar to the eigenvalue integral of a matrix model. Namely, Q-functions correspond to a potential term for the eigenvalues and $\prod_{k<l}(x_k-x_l)(e^{2\pi x_k}-e^{2\pi x_l})$ can be interpreted as a modified Vandermonde factor. It is intriguing to note that this modified Vandermonde factor is a hybrid of the ordinary Vandermonde factor for the Hermitian matrix model, $\prod_{k<l}(x_k-x_l)^2$, and the generalized Vandermonde factor for the unitary matrix model and the Chern-Simons matrix model [29], which is essentially given by $\prod_{k<l}(e^{2\pi x_k}-e^{2\pi x_l})^2$. This resemblance to a matrix model strongly suggests that the semi-classical limit for the scalar product can be analyzed by applying the method of large N expansion familiar for matrix models. It turns out, however, that the separated variables we deal with here and the eigenvalues of a matrix model behave somewhat differently and one must be very careful in adapting such a method. This subject will be discussed further in the final section.

5 Discussions

In this paper, applying the method of separation of variables à la Sklyanin, we have derived a multiple contour integral representation for the scalar products between an on-shell and an off-shell Bethe states of the XXX spin 1/2 chain with periodic boundary

condition. Although a compact determinant expression for such a scalar product was discovered already a decade ago [12] and various variants have been discussed since then, our novel integral formula should be useful for a number of purposes. One such area is the study of the semi-classical behavior of the scalar products. In particular, this is quite important for the comparison of the three point functions in super Yang-Mills theory and the corresponding dual string theory. Already results have been obtained in this regard [6,16,17] starting from the determinantal form of the scalar product. However, the procedure through which these results are obtained is ingenious but not quite systematic. The purpose of our present work is largely to improve on this situation.

There are indeed apparent advantages for our formula for such a purpose. Firstly, almost by definition, the SoV method we employed guarantees that the factorization of the dependence on the basic variables occur. Such a factorization is not realized in the determinant formula and indeed it took some non-trivial steps to obtain such a structure in [16]. A further merit of the inherent factorization is that the factorized dependence on the inhomogeneous parameters θ_i may be quite useful in applying the so-called Θ-morphism operation [7,8], which could be a key for understanding the loop effects.

Another advantage of our formula is that, as was shown in section 4, it can be put into a form reminiscent of the integral over the eigenvalues of a matrix model. An intriguing fact is that the measure factor of our integral formula looks like a hybrid of that of a hermitian matrix model and a unitary matrix model. The semi-classical limit of interest corresponds to the large size limit of the matrix and various techniques developed in the past may be utilized. However, preliminary investigation indicates that there are certain important differences between our integral and the matrix integral. For instance, in contrast to the usual matrix model eigenvalues, the separated variables x_k's do not necessarily condense. It appears that careful analysis for the various regions of x_k's is required in order to extract the semi-classical contributions systematically. Such an analysis will be presented elsewhere [30].

Apart from the semi-classical limit, our formalism should be useful for the general development of solvable models. An immediate application may be to the SU(3) spin chain, for which the SoV method is available [31,32]. It is of interest to see what modification from the SU(2) case would be needed to obtain a nice integral formula for the scalar product for such a system.

Before ending this article, we should perhaps emphasize, alongside with Sklyanin, that the method of SoV, or functional Bethe ansatz as it was originally called, is in a sense the most fundamental and general of all the Bethe ansatz methods and hence should

be applicable to systems to which the other methods are not easily applicable. In this sense, we hope that our analysis will prove to be useful for future development of the SoV method.

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A Proof of theorem 1

In this appendix, we shall provide a proof of the theorem 1, which gives the action of the normal-ordered operators $:A_\epsilon(\hat{x}_i):_L$ and $:D_\epsilon(\hat{x}_i):_L$ on the SoV bra $\langle \dots, x_i, \dots |$. Essentially, what follows is a pedagogical adaptation of the argument given in [33].

The proof is by mathematical induction in the number of sites L. Begin with the $L = 1$ case. The operators $A(u), B(u), C(u), D(u)$ are given by[15]

$$A(u) = u - \theta_1 + iS^2 + i\epsilon S^+ , \qquad B(u) = \epsilon(u - \theta_1 - iS^3) + iS^- , \tag{A.1}$$

$$C(u) = -\epsilon(u - \theta_1 + iS^3) + iS^+ , \qquad D(u) = u - \theta_1 - iS^3 - i\epsilon S^- . \tag{A.2}$$

By solving $B(\hat{x}_1) = 0$ for $\hat{x}_1$ and substituting it into $A(u)$ and $D(u)$, we get

$$\hat{x}_1 = \theta_1 + iS^3 - i\epsilon^{-1}S^- = \begin{pmatrix} \theta_1 + \frac{i}{2} & 0 \\ -i\epsilon^{-1} & \theta_1 - \frac{i}{2} \end{pmatrix} , \tag{A.3}$$

$$:A(\hat{x}_1):_L = 2iS^3 - i\epsilon^{-1}S^- + i\epsilon S^+ = i \begin{pmatrix} 1 & \epsilon \\ -\epsilon^{-1} & -1 \end{pmatrix} , \tag{A.4}$$

$$:D(\hat{x}_1):_L = -i(\epsilon + \epsilon^{-1})S^- = -i(\epsilon + \epsilon^{-1}) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{A.5}$$

Since $\hat{x}_1$ is lower triangular, its eigenvalues are read off as $\theta_1 \pm \frac{i}{2}$ and the corresponding normalized eigenbras $\langle \pm |$ are given by $\langle + | = (1, 0), \langle - | = (1, \epsilon)/\sqrt{1 + \epsilon^2}$. Then, we can compute the action of $:A(\hat{x}_1):_L$ and $:D(\hat{x}_1):_L$ explicitly and get

$$\langle + |:A(\hat{x}_1):_L = i\sqrt{1 + \epsilon^2}\langle - | , \quad \langle - |:A(\hat{x}_1):_L = 0 , \tag{A.6}$$

$$\langle + |:D(\hat{x}_1):_L = 0 , \qquad \langle - |:D(\hat{x}_1):_L = -i\sqrt{1 + \epsilon^2}\langle + | . \tag{A.7}$$

[15] We have dropped the subscript 1 for S^i for simplicity.

This is precisely what the theorem says for $L = 1$.

Next, assume that the formulas hold for up to $L = N$ and consider $L = N + 1$ case. The monodromy matrix for $L = N + 1$ is given by

$$\Omega_{N+1} = KL_1 \cdots L_N L_{N+1}\,, \tag{A.8}$$

where K is the twisting matrix given in (**). Now in order to split this into the monodromy matrix at the Nth level and the subsequent action of the Lax operator at the $(N+1)$th step, it is technically convenient to introduce in the final Lax operator a twisting matrix of a similar form, which we denote as

$$\tilde{K} \equiv \begin{pmatrix} 1 & \eta \\ -\eta & 1 \end{pmatrix}, \tag{A.9}$$

with η being an arbitrary parameter, just like ϵ. Therefore we write $\Omega_{N+1} = \tilde{\Omega}_N L'_{N+1}$, where

$$\tilde{\Omega}_N = (K\tilde{K}^{-1})\widetilde{L}_1\widetilde{L}_2 \cdots \widetilde{L}_N\,, \qquad K\tilde{K}^{-1} = \begin{pmatrix} 1 & \epsilon - \eta \\ -(\epsilon - \eta) & 1 \end{pmatrix}, \tag{A.10}$$

$$\widetilde{L}_i = \tilde{K}L_i\tilde{K}^{-1}\,, \qquad L'_{N+1} = \tilde{K}L_{N+1}\,. \tag{A.11}$$

Since the conjugation by $\tilde{K}$ does not affect the structure of the algebra, we may regard $\tilde{\Omega}_N$ as the monodromy matrix for $L = N$ for which the theorem holds. We now write the matrix elements of $\tilde{\Omega}_N$ and L'_{N+1} as

$$\tilde{\Omega}_N(u) = \begin{pmatrix} \widetilde{A}_N & \tilde{B}_N \\ \widetilde{C}_N & \widetilde{D}_N \end{pmatrix}, \qquad L'_{N+1} = \begin{pmatrix} a_{N+1} & b_{N+1} \\ c_{N+1} & d_{N+1} \end{pmatrix}, \tag{A.12}$$

and compute $\Omega_{N+1} = \begin{pmatrix} A_{N+1} & B_{N+1} \\ C_{N+1} & D_{N+1} \end{pmatrix}$. Then B_{N+1} operator is given by

$$B_{N+1}(u) = \widetilde{A}_N(u)b_{N+1}(u) + \tilde{B}_N(u)d_{N+1}(u)\,. \tag{A.13}$$

Let us now write the SoV basis of bras for $L = N+1$ as $\langle x_1, \ldots, x_N; y|$. By the hypothesis of the induction, $\tilde{B}_N(u)$ is diagonal in this basis and also b_{N+1}, which acts only on the $(L+1)$th site, is diagonal. Explicitly, we have

$$\langle x_1, \ldots, x_N; y|\tilde{B}_N(u) = (\epsilon - \eta)\prod_{i=1}^{N}(u - x_i)\langle x_1, \ldots, x_N; y|\,, \tag{A.14}$$

$$\langle x_1, \ldots, x_N; y|b_{N+1}(u) = \eta(u - y)\langle x_1, \ldots, x_N; y|\,. \tag{A.15}$$

We may now compute the action of $\tilde{B}_{N+1}(u)$ at $u = \hat{x}_k$ and $u = \hat{y}$, where $\hat{y}$ is the root of $b_{N+1}(\hat{y}) = 0$ given by $\hat{y} = \theta_{N+1} + iS^3_{N+1} - i\eta^{-1}S^-_{N+1}$. Since $\tilde{B}_N(\hat{x}_k)$ and $b_{N+1}(\hat{y})$ vanishes on this state, we get

$$\langle x_1, \dots, x_N; y|B_{N+1}(\hat{x}_k) = \langle x_1, \dots, x_N; y|\widetilde{A}_N(\hat{x}_k)b_{N+1}(\hat{x}_k)\,, \tag{A.16}$$

$$\langle x_1, \dots, x_N; y|B_{N+1}(\hat{y}) = \langle x_1, \dots, x_N; y|\tilde{B}_N(\hat{y})d_{N+1}(\hat{y})\,. \tag{A.17}$$

The RHS can be easily computed since $\widetilde{A}_N(\hat{x}_k)$ shifts x_k by $-i$, while $d_{N+1}(\hat{y})$ shifts y by $+i$, with certain known factors multiplied. In this way, we obtain the formulas

$$\langle x_1, \dots, x_N; y|B_{N+1}(\hat{x}_k) = \eta(x_k - \eta)Q^+_{\boldsymbol{\theta}}(x_k)\langle \dots, x_k - i, \dots; y|\,, \tag{A.18}$$

$$\langle x_1, \dots, x_N; y|B_{N+1}(\hat{y}) = (\epsilon - \eta)(y - \theta_{N+1} - i/2)\prod_{k=1}^{N}(y - x_k)\langle \dots, x_k, \dots; y + i|\,. \tag{A.19}$$

Having understood the action of B_{N+1} at $u = \hat{x}_k, \hat{y}$ on the SoV basis, we now wish to deduce the spectrum of $B_{N+1}(u)$ using this information. Let $|\Phi\rangle$ be the state which diagonalizes $B_{N+1}(u)$. Then by taking the inner product with the above two equations, we obtain

$$\beta(x_k)\Phi(x_1, \dots, x_n; y) = \eta(x_k - \eta)Q^+_{\boldsymbol{\theta}}(x_k)\Phi(\dots, x_k - i, \dots; y)\,, \tag{A.20}$$

$$\beta(y)\Phi(x_1, \dots, x_N; y) = (\epsilon - \eta)(y - \theta_{N+1} - i/2)\prod_{k=1}^{N}(y - x_k)\Phi(\dots, x_k, \dots; y + i)\,, \tag{A.21}$$

where $\Phi(x_1, \dots, x_N; y) \equiv \langle x_1, \dots, x_N; y|\Phi\rangle$ and we have denoted the eigenvalue of $B_{N+1}(u)$ by $\beta(u)$. Now to simplify the analysis of the spectrum, it is convenient to extract a factor $\rho(x_1, \dots, x_N; y)$ from $\Phi(x_1, \dots, x_N; y)$ in the manner

$$\Phi(x_1, \dots, x_N; y) = \rho(x_1, \dots, x_N; y)\Psi(x_1, \dots, x_N; y) \tag{A.22}$$

where $\rho(x_1, \dots, x_N; y)$ satisfies the first order difference equations

$$\rho(x_1, \dots, x_N; y) = \eta(x_k - \eta)\rho(\dots, x_k - i, \dots; y)\,, \tag{A.23}$$

$$\rho(x_1, \dots, x_N; y) = (\epsilon - \eta)\prod_{i=1}^{N}(y - x_k)\rho(\dots, x_k, \dots; y + i)\,. \tag{A.24}$$

One can easily verify that the solution to these equations is unique[16] up to an overall constant. Now with such a factor removed, the reduced wave function Ψ satisfies the

[16] The uniqueness is guaranteed by the finiteness of the spectrum of $\hat{x}_k$ and $\hat{y}$. One can construct the solution ρ by starting from the end of the spectrum.

equations

$$\beta(x_k)\Psi(x_1,\ldots,x_N;y) = Q^+_{\boldsymbol{\theta}}(x_k)\Psi(\ldots,x_k-i,\ldots;y)\,, \tag{A.25}$$

$$\beta(y)\Psi(x_1,\ldots,x_N;y) = (y-\theta_{N+1}-i/2)\Psi(\ldots,x_k,\ldots;y+i)\,. \tag{A.26}$$

It turns out that we can drastically simplify these equations by assuming the factorized form[17] for Ψ, namely

$$\Psi(x_1,\ldots,x_N;y) = \chi(y)\prod_{k=1}^{N}\xi_k(x_k)\,. \tag{A.27}$$

The equations for Ψ then get reduced to the following equations for each factor

$$\beta(x)\xi_k(x) = Q^+_{\boldsymbol{\theta}}(x)\xi_k(x-i)\,, \qquad x\in\left\{\theta_k-\frac{i}{2},\theta_k+\frac{i}{2}\right\}\,, \tag{A.28}$$

$$\beta(x)\chi(x) = (y-\theta_{N+1}-i/2)\chi(x+i)\,, \qquad x\in\left\{\theta_{N+1}-\frac{i}{2},\theta_{N+1}+\frac{i}{2}\right\}\,. \tag{A.29}$$

Note that we have used the induction hypothesis that the spectrum of each x_k is two-valued as above.

The rest of the analysis is elementary. First consider the equation (A.28) and set $x=\theta_k-\frac{i}{2}$. Then due to the presence of the factor $Q^+_{\boldsymbol{\theta}}(x)$ the RHS vanishes and hence we must have $\beta(\theta_k-\frac{i}{2})\xi_k(\theta_k-\frac{i}{2})=0$. If $\xi_k(\theta_k-\frac{i}{2})\neq 0$, then $\beta(\theta_k-\frac{i}{2})$ must vanish and $\theta_k-\frac{i}{2}$ is in the spectrum. On the other hand suppose $\xi_k(\theta_k-\frac{i}{2})=0$. Then $\xi_k(\theta_k+\frac{i}{2})$ cannot vanish since otherwise the whole wave function vanishes. Now set $x=\theta_k+\frac{i}{2}$ in (A.28). Then the RHS vanishes and so must the LHS, *i.e.* $\beta(\theta_k+\frac{i}{2})\xi_k(\theta_k+\frac{i}{2})=0$. This leads to $\beta(\theta_k+\frac{i}{2})=0$ and hence $x=\theta_k+\frac{i}{2}$ is in the spectrum. Similar arguments for (A.29) tells us that $\theta_{N+1}\pm\frac{i}{2}$ are in the spectrum. Thus, for $L=N+1$, we continue to have the same set of spectrum as stated in the theorem.

From this analysis we learn that the finite discrete nature of the spectrum is due to two reasons. One is that the operators $:A(x_k):_L$ and $:D(x_k):_L$ are essentially exponentials of the momentum operator and hence they induce a finite shift in x_k. The second ingredient is the presence of the prefactor $Q^+_{\boldsymbol{\theta}}(x)$. Since it vanishes at finite discrete values of x, the shifting must end after a finite number of steps, in the present case just one.

What remains is the determination of the constant of proportionality in the action of the operators $:A(x_k):_L$ and $:D(x_k):_L$. As there are only a finite number of states, such a constant can be adjusted rather freely by the change of the normalization of states. Nonetheless, there is a certain constraint coming from the following non-linear relations:

[17] This does not miss any solution since the solution is unique.

$$:A_{N+1}(\hat{x}_k):_L:D_{N+1}(\hat{x}_k):_L = (1+\epsilon^2)\prod_{i=1}^{N+1}(\hat{x}_k-\theta_i+i/2)(\hat{x}_k-\theta_i-3i/2)\,, \tag{A.30}$$

$$:D_{N+1}(\hat{x}_k):_L:A_{N+1}(\hat{x}_k):_L = (1+\epsilon^2)\prod_{i=1}^{N+1}(\hat{x}_k-\theta_i-i/2)(\hat{x}_k-\theta_i+3i/2)\,. \tag{A.31}$$

These relations can be obtained in the following way. From the commutation relations between $:A_{N+1}(\hat{x}_k):_L$, $:D_{N+1}(\hat{x}_k):_L$ and $\hat{x}_k$, one can show

$$:A_{N+1}(\hat{x}_k):_L:\, D_{N+1}(\hat{x}_k):_L = \det{}_q\Omega_{N+1}(\hat{x}_k-i/2)\,, \tag{A.32}$$

$$:D_{N+1}(\hat{x}_k):_L:A_{N+1}(\hat{x}_k):_L = \det{}_q\Omega_{N+1}(\hat{x}_k+i/2)\,, \tag{A.33}$$

where $\det_q\Omega_{N+1}(u)$ is the so-called quantum determinant[18] , which is a central element of the Yang-Baxter exchange algebra. Then by using the co-multiplication rule, $\det_q(AB) = \det_q A\,\det_q B$, one can explicitly compute the RHS and obtain the relations (A.30) and (A.31). The constant of proportionality chosen in the theorem is compatible with these relations and also to the explicit equations for $L = 1$ case shown in (A.6) and (A.7) obtained for unit-normalized states. This completes the proof of the theorem.

B Relation to Izergin's determinant formula

In this appendix, we give a direct proof that a slight generalization of our new integral expression is equivalent to the Izergin's determinant formula [27] for the domain wall partition function (DWPF) which appears in the six-vertex model. From this DWPF, the original scalar product of our interest can be obtained by sending an appropriate subset of rapidities to infinity as well as requiring half of the remainder to be on-shell.

We begin with the domain wall partition function, which is defined as follows:

$$Z_L(\boldsymbol{w}|\boldsymbol{\theta}) \equiv \langle\downarrow^L|\prod_{i=1}^{L}B(w_i)|\uparrow^L\rangle\,. \tag{B.1}$$

Note that the number of B operators is equal to the number of sites L and the rapidities $\boldsymbol{w}$ are not restricted to an on-shell configuration. In [27], Izergin gave a determinant expression for this quantity, which reads

$$Z_L(\boldsymbol{w}|\boldsymbol{\theta}) = \frac{\prod_{i,j=1}^{L}(w_i-\theta_j+\frac{i}{2})(w_i-\theta_j-\frac{i}{2})}{\prod_{i<j}(w_i-w_j)(\theta_j-\theta_i)}\det\left(\frac{i}{(w_i-\theta_j+\frac{i}{2})(w_i-\theta_j-\frac{i}{2})}\right)_{1\le i,j\le L}. \tag{B.2}$$

[18]For a detailed account of the quantum determinant, we refer the reader to [1] and [33].

In what follows, we shall show that this is equal to the multiple integral formula of the form

$$\langle\downarrow^L|\prod_{i=1}^{L}B(w_i)|\uparrow^L\rangle = i^L\prod_{i<j}(\theta_i-\theta_j)(\theta_i-\theta_j+i)(\theta_i-\theta_j-i) \times\prod_{j=1}^{L}\oint_{\mathcal{C}_j}\frac{dx_j}{2\pi i}\prod_{i<j}(x_i-x_j)\prod_{i=1}^{L}\frac{Q_{\boldsymbol{w}}(x_i)}{Q^+_{\boldsymbol{\theta}}(x_i)Q^-_{\boldsymbol{\theta}}(x_i)}\,. \tag{B.3}$$

First, we shall transform the Izergin's formula to a form more convenient for comparison with the integral expression. By a simple decomposition, the determinant in (B.2) can be rewritten as a determinant of the difference of two matrices:

$$\det\left(\frac{i}{(w_i-\theta_j+\frac{i}{2})(w_i-\theta_j-\frac{i}{2})}\right)_{1\le i,j\le L} = \det(M^-_{ij}-M^+_{ij})_{1\le i,j\le L}\,, \tag{B.4}$$

$$M^\pm_{ij} = \frac{1}{w_i-\theta_j\pm i/2}\,. \tag{B.5}$$

Then from the definition of the determinant, we can expand the RHS of (B.5) as

$$\begin{aligned}\det(M^-_{ij}-M^+_{ij})_{1\le i,j\le L} &= \sum_{\sigma\in P_L}(-1)^\sigma(M^-_{1\sigma(1)}-M^+_{1\sigma(1)})\cdots(M^-_{L\sigma(L)}-M^+_{L\sigma(L)})\\ &= \sum_{\epsilon_i=\pm}(-1)^{n_+}\sum_{\sigma\in P_L}(-1)^\sigma M^{\epsilon_1}_{1\sigma(1)}\cdots M^{\epsilon_L}_{L\sigma(L)}\,,\end{aligned} \tag{B.6}$$

where n_+ is the number of $+$'s in the set $\{\epsilon_i\}$ and the sign $(-1)^{n_+}$ is produced upon expanding the product. Now by using the definition of determinant again to re-express each summand back as a determinant, we obtain

$$\det(M^-_{ij}-M^+_{ij})_{1\le i,j\le L} = \sum_{\epsilon_i=\pm}(-1)^{n_i}\det(M^{\epsilon_i}_{ij})_{1\le i,j\le L}\,. \tag{B.7}$$

At this point, one can apply the Cauchy's determinant identity,

$$\det\left(\frac{1}{x_i-y_j}\right)_{1\le i,j\le L} = \frac{\prod_{1\le i,j\le L}(x_i-x_j)(y_j-y_i)}{\prod_{i,j=1}^{L}(x_i-y_j)}\,, \tag{B.8}$$

to each term $\det(M^{\epsilon_i}_{ij})_{1\le i,j\le L} = \det((w_i-(\theta_j-\epsilon_i i/2))^{-1})_{1\le i,j\le L}$. Putting altogether the determinant (B.5) in the Izergin's formula can be expressed as

$$\det\left(\frac{i}{(w_i-\theta_j+\frac{i}{2})(w_i-\theta_j-\frac{i}{2})}\right)_{1\le i,j\le L} = \frac{\prod_{1\le i,j\le L}(w_i-w_j)(\theta_j-\theta_i-i(\epsilon_j-\epsilon_i)/2)}{\prod_{i,j=1}^{L}(w_i-\theta_j+i\epsilon_j/2)}\,. \tag{B.9}$$

Substituting it into (B.2), Izergin's formula is finally transformed into the expression

$$Z_L(\boldsymbol{w}|\boldsymbol{\theta}) = \sum_{\epsilon_i=\pm} (-1)^{n_+} \prod_{i,j=1}^{L} (w_i - (\theta_j + i\epsilon_j/2)) \prod_{1\le i,j\le L} \frac{(\theta_i - \theta_j - i(\epsilon_i - \epsilon_j)/2)}{\theta_i - \theta_j}\,, \tag{B.10}$$

which is no longer of a determinant form.

Now we are ready to prove its equivalence to the multiple integral (B.3). This is done essentially by explicitly performing the contour integrals using the residue formula. By picking up the contributions from the zeros of the functions $Q^{\pm}_{\boldsymbol{\theta}}(x_i)$ in the denominator, the integral is evaluated as

$$\begin{aligned}&\prod_{i<j}(\theta_i - \theta_j)(\theta_i - \theta_j + i)(\theta_i - \theta_j - i)\\ &\times \sum_{\epsilon_i=\pm}(-1)^{n_+} \prod_{1\le i,j\le L} \frac{(\theta_i - \theta_j + i(\epsilon_i - \epsilon_j)/2)}{(\theta_i - \theta_j)^2(\theta_i - \theta_j + \epsilon_i)(\theta_i - \theta_j - \epsilon_j)} \prod_{i,j=1}^{L} (w_i - (\theta_j + i\epsilon_j/2))\,.\end{aligned} \tag{B.11}$$

Now note the following relation, which can be checked for every pair (ϵ_i, ϵ_j), with $\epsilon_i = \pm 1$:

$$\frac{(\theta_i - \theta_j + i)(\theta_i - \theta_j - i)}{(\theta_i - \theta_j + i\epsilon_i)(\theta_i - \theta_j - i\epsilon_j)} = \frac{(\theta_i - \theta_j - i(\epsilon_i - \epsilon_j)/2)}{(\theta_i - \theta_j + i(\epsilon_i - \epsilon_j)/2)}\,. \tag{B.12}$$

Using this formula, the expression (B.11) can be simplified into

$$\sum_{\epsilon_i=\pm}(-1)^{n_+} \prod_{i,j=1}^{L} (w_i - (\theta_j + i\epsilon_j/2)) \prod_{1\le i,j\le L} \frac{(\theta_i - \theta_j - i(\epsilon_i - \epsilon_j)/2)}{\theta_i - \theta_j}\,. \tag{B.13}$$

This is exactly the same as (B.10), proving the assertion.

As already stated, the original scalar product of our interest can be obtained from this domain wall partition function through certain manipulations. First, by sending $L-n$ of the L rapidities to infinity, thereby decoupling them, one obtains the partial domain wall partition function with n rapidities $\boldsymbol{z}$ (2.18) in the following way:

$$Z^{\mathrm{pDWPF}}(\boldsymbol{z}|\boldsymbol{\theta}) = \frac{1}{(L-n)!} \lim_{\{w_1,\dots,w_{L-n}\}\to\infty} \left(\frac{Z_L(\boldsymbol{z} \cup \{w_1, \dots, w_{L-n}\}|\boldsymbol{\theta})}{i w_1^{L-1} \cdots i w_{L-n}^{L-1}} \right)\,. \tag{B.14}$$

If we now set $n = 2M$ and $\boldsymbol{z} = \boldsymbol{u} \cup \boldsymbol{v}$, where either $\boldsymbol{u}$ or $\boldsymbol{v}$ are on-shell, we recover the original scalar product $\langle \uparrow^L | \prod_{i=1}^{M} C(v_i) \prod_{j=1}^{M} B(u_j) | \uparrow^L \rangle$. On the other hand, if we apply the same manipulations to the integral formula (B.2), we obtain the multiple integral formula for the scalar product (4.38). This proves the equivalence of our formula and the determinant formula derived by Foda and Wheeler [26].

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